Journal of Statistical Physics, Vol. 77, Nos. 3/4, 1994

A Note on the Projecton of Gibbs Measures

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Received December 16, 1993; final April 26, 1994

We give an example of a projection which maps two Gibbs measures for the same interaction into Gibbs measures for different interactions. As a corollary we find a case where by decimation a non-Gibbsian measure is transformed into a Gibbs measure.

KEY WORDS: Gibbs measures; non-Gibbsian measures; decimation transformation; projected measures.

1. INTRODUCTION

In recent years there has emerged an increasing interest in the study of restrictions of Gibbs measures, coming from both renormalization-group theory and interacting particle systems. In renormalization-group theory one considers, for instance, decimation (that is, restriction to a sublattice of the same dimension as the original lattice) applied to the pure phase(s) of the system described by a given interaction. In probabilistic cellular automata (PCA) the study of stationary states which are projections (that is, restrictions to a lower-dimensional sublattice) of Gibbs measures is of particular interest. In both classes of these examples one starts with the Gibbs measures for the given interaction. After the application of the respective transformation the result in all known examples is one of the following two cases: The decimation or projection of Gibbs measures for the same interaction yields either Gibbs measures for which the interactions coincide, or else it produces non-Gibbsian measures. Only these two possibilities occur, for instance, in the case of the stationary states of various PCA where strictly positive transition probabilities are given, and

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also for general block-spin transformations on lattice spin systems (for details see ref. 1, Theorem 1; ref. 2, Theorem 3.4). Also Schonmann's example (ref. 3, Proposition 2) follows this pattern, since the projection on the line of the pure phases of the two-dimensional nearest-neighbor ferromagnetic Ising model in the subcritical regime leads to non-Gibbsian measures.

Here we provide an example similar to Schonmann's, but in which the projection maps two Gibbs measures for the same interaction into Gibbs measures for different interactions. At the same time our example implies a case where by decimation a non-Gibbsian measure transforms into a Gibbs measure.

2. MAIN RESULT

We consider here spins assuming the values -1 and $+1$, living on $\mathbb{Z}^2$ and its various sublattices. The identification $\mathbb{Z} = \mathbb{Z} \otimes \{0\}$ will be made, and we will call the corresponding configuration spaces $\Omega = \{-1, +1\}^{\mathbb{Z}^2}$ and $\Omega_1 = \{-1, +1\}^{\mathbb{Z}}$. Next we introduce the family of one-dimensional sublattices $l\mathbb{Z} = \{x \in \mathbb{Z}: x \bmod l = 0\}$, and the associated configuration spaces $\Omega_l = \{-1, +1\}^{l\mathbb{Z}}$. Denote by μ^+ and μ^- the $+$ and $-$ phase, respectively, of the two-dimensional ferromagnetic nearest-neighbor Ising model (above the critical inverse temperature β_c), and by ν_l^+ and ν_l^-, respectively, their projections onto $l\mathbb{Z}$. Note that the measures ν_l^+ and ν_l^- can also be thought of as being the decimated measures of ν_1^+ and ν_1^-, onto $l\mathbb{Z}$.

For completeness we first give some notation and results concerning cluster expansions (more details can be found in ref. 4). For a sequence of side lengths $L \uparrow \infty$, we consider the square boxes

$$V \equiv V_L = \{i = (x, y) \in \mathbb{Z}^2 : -L \leqslant x, y \leqslant L\}$$

and their one-dimensional segments

$$\Lambda_l = V \cap l\mathbb{Z} = \{i = (x, y) \in V: y = 0, x \bmod l = 0\}$$

The subvolume W_l is defined by

$$W_l = V \backslash \Lambda_l$$

With $+$ boundary conditions, the energy of a configuration σ on V is given by

$$H_V^+(\sigma) = -\sum_{\langle ij \rangle \cap V \neq \varnothing} (\sigma_i \sigma_j - 1) \tag{1}$$

where the sum is over all nearest-neighbor pairs $\langle ij \rangle$, $i \in V$ or $j \in V$. We put $\sigma_i = +1$ for all $i \notin V$.

Let μ_V^+ be the corresponding Gibbs measure on V at inverse temperature β, i.e.,

$$\mu_V^+(\sigma) = Z_V^{-1}(\beta) \exp[-\beta H_V^+(\sigma)]$$

where $Z_V(\beta)$ is the usual normalizing partition function. We want to study the projection $\nu_{\Lambda_l}^+$ of μ_V^+, i.e., the restriction of μ_V^+ to the segment Λ_l. For a configuration ξ on Λ_l, $\nu_{\Lambda_l}^+$ has weights

$$\begin{aligned} \nu_{\Lambda_l}^+(\xi) &= \mu_V^+(\sigma = \xi \text{ on } \Lambda_l) \\ &= \sum_{\sigma_i = \pm 1, i \in W_l} \mu_V^+(\sigma)|_{\sigma = \xi \text{ on } \Lambda_l} \end{aligned} \tag{2}$$

Obviously, for any finite L and β, $\nu_{\Lambda_l}^+$ is a Gibbs measure for some Hamiltonian

$$\mathscr{H}_{\Lambda_l}(\xi) = -\log \nu_{\Lambda_l}^+(\xi)$$

Its conditional probability distribution at the origin is

$$\nu_{\Lambda_l}^+(\sigma_o = \xi_0 \mid \sigma = \xi \text{ on } \Lambda_l \backslash o) = \frac{1}{1 + \exp[h_{\Lambda_l}^+(\xi)]} \tag{3}$$

Here,

$$h_{\Lambda_l}^+(\xi) = -\log \frac{\nu_{\Lambda_l}^+(\xi)}{\nu_{\Lambda_l}^+(\xi^o)} \tag{4}$$

is the energy difference $\mathscr{H}_{\Lambda_l}^+(\xi) - \mathscr{H}_{\Lambda_l}^+(\xi^o)$ (or relative energy) for flipping the spin at the origin, and ξ^o is the configuration defined by

$$\xi_x^o = \begin{cases} \xi_x & \text{if } x \neq 0 \\ -\xi_o & \text{if } x = 0 \end{cases}$$

Note also that for $l > 1$

$$\exp[-h_{\Lambda_l}^+(\xi)] = \frac{Z_{W_l}^{+;\xi}(\beta)}{Z_{W_l}^{+;\xi^o}(\beta)} \tag{5}$$

where

$$Z_{W_l}^{+;\xi}(\beta) = \sum_{\sigma_i = \pm 1, i \in W_l} \exp[-\beta H_{W_l}^{+;\xi}(\sigma)] \tag{6}$$

$$H_{W_l}^{+;\xi}(\sigma) = -\sum_{\langle ij \rangle \cap W_l \neq \varnothing} (\sigma_i \sigma_j - 1)|_{\sigma = \xi \text{ on } \Lambda_l} \tag{7}$$

are the partition function and the Hamiltonian, respectively, for the volume W_l with $+$ boundary conditions outside and ξ boundary conditions on Λ_l. Similar definitions can be given starting from $-$ boundary conditions on V, changing all superscripts $+$ into $-$.

We will represent the configurations on W_l by sets of disjoint closed contours, as is usually done for $+$ boundary conditions. Let Γ_{W_l} denote the set of all closed contours on W_l. If a configuration σ is represented by a set of contours $\{\gamma_\alpha\}_{\alpha=1}^n \subset \Gamma_{W_l}$, then the Hamiltonian $H_{W_l}^+$ corresponding to $+$ boundary conditions is given by

$$H_{W_l}^+(\sigma) = 2 \sum_{\alpha=1}^{n} |\gamma_\alpha|$$

where $|\gamma_\alpha|$ is the length of γ_α.

For a given contour γ and $\xi \in W_l$, define $c_l((\gamma, \xi)$ as the number of edges of γ touching some site $x \in l\mathbb{Z}$ for which $\xi_x = -1$. It is not hard to see then that

$$-h_{\Lambda_l}^+(\xi) = 4\beta\xi_o + \log \frac{\tilde{Z}_{W_l}^{+,\xi}(\beta)}{\tilde{Z}_{W_l}^{+,\xi^o}(\beta)} \tag{8}$$

where

$$\tilde{Z}_{W_l}^{+,\xi}(\beta) = 1 + \sum_{n=1} \frac{1}{n!} \sum_{\substack{\gamma_1 \cdots \gamma_n \in \Gamma_{W_l} \\ \text{disjoint}}} \prod_{\alpha=1}^{n} z_\xi(\gamma_\alpha)$$

and

$$z_\xi(\gamma) = \exp[-2\beta\, |\gamma| + 4\beta c_l(\gamma, \xi)]$$

We use the technique of cluster expansions to calculate the ratio of the two partition functions. We obtain that

$$-h_{\Lambda_l}^+(\xi) = 4\beta\xi_o + \sum_{n=1} \frac{1}{n!} \sum_{\gamma_1 \cdots \gamma_n \in \Gamma_{W_l}} \psi_n^T(\gamma_1 \cdots \gamma_n) \times \left[\prod_{\alpha=1}^{n} z_\xi(\gamma_\alpha) - \prod_{\alpha=1}^{n} z_{\xi_o}(\gamma_\alpha) \right] \tag{9}$$

where

$$\psi_n^T(\gamma_1 \cdots \gamma_n) = \sum_{\substack{C \text{ connected graphs} \\ \text{with } n \text{ vertices}}} \prod_{\substack{(\alpha\alpha') \text{ is an} \\ \text{edge of } C}} \psi(\gamma_\alpha, \gamma_{\alpha'})$$

$$\psi(\gamma_\alpha, \gamma_{\alpha'}) = \begin{cases} 0 & \text{if } \gamma_\alpha, \gamma_{\alpha'} \text{ are disjoint} \\ -1 & \text{otherwise} \end{cases} \tag{10}$$

Theorem 1. For any $l \geqslant 3$ there exists $\beta_c \leqslant \beta_l < \infty$ such that for all $\beta \geqslant \beta_l$ the projections ν_l^+ and ν_l^- are Gibbs measures with respect to two physically nonequivalent, absolutely summable interactions on Ω_l.

Proof. We use the low-temperature expansion for the relative energy of the projected measures as described above (see also ref. 4). Uniform convergence of the expansion would imply the existence of a continuous version of conditional probabilities determined by the relative energy. This then would guarantee the existence of an absolutely summable interaction for which the measure would be Gibbsian (ref. 2, Theorem 2.12; see also references therein).

Let us denote by $h_l^+(\xi)$ the relative energy for ν_l^+ of a spinflip at an arbitrary site. We make an expansion for $h_l^+(\xi)$ in terms of contours on (the dual lattice of) $\mathbb{Z}^2 \backslash l\mathbb{Z}$. The weight of a contour γ is

$$z_\xi(\gamma) = \exp[-2\beta\,|\gamma| + 4\beta c(\gamma, \xi)]$$

where $c(\gamma, \xi)$ is the number of edges of γ touching some site $x \in l\mathbb{Z}$ for which $\xi_x = -1$, and $|\gamma|$ denotes the length of the contour γ. Uniform convergence is obtained at sufficiently low temperatures if there exists some $K_l > 0$ such that for any configuration ξ the weight $z_\xi(\gamma) \leqslant \exp(-2K_l\beta\,|\gamma|)$. To cope with the worst that can happen we choose $\xi_x = -1$ for all $x \in l\mathbb{Z}$, as then $c(\gamma, \xi)$, for fixed γ, reaches its maximum. Figure 1 shows the "worst" type of contour (for $l = 3$), that is, the one which visits as many minuses as possible, given its length. Clearly, for $K_l = (l-2)/(l+1)$ the inequality

$$|\gamma| - 2c(\gamma, \xi) \geqslant K_l\,|\gamma|$$

holds for $\xi_x = -1$ and the "worst" contour, and therefore for all ξ and all contours. Thus we have

$$z_\xi(\gamma) \leqslant \exp(-2\beta K_l\,|\gamma|)$$

which is sufficient for uniform convergence. Moreover, as l grows, K_l

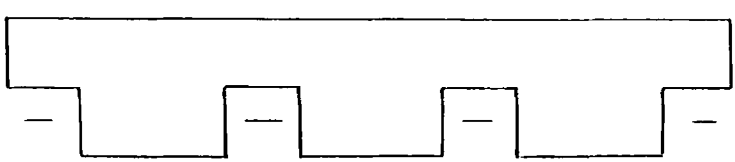

Fig. 1. The "worst" type of contour ($l = 3$).

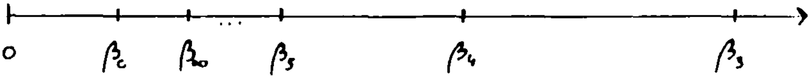

Fig. 2. Range of convergence of the expansion.

becomes larger, thus the uniform convergence extends to wider temperature ranges. These temperature ranges are delimited by the values

$$\beta_l = \frac{\beta_\infty}{K_l} = \frac{l+1}{l-2}\,\beta_\infty$$

where β_∞ is the threshold temperature for the cluster expansion on $\mathbb{Z}^2$ (see Fig. 2). Obviously, $h_l^+(\xi) = h_l^-(-\xi)$ for any $\xi \in \Omega_l$, but as also can be seen from the expansion, the functions h_l^+ and h_l^- are not even in ξ. In particular, they are different and the associated Hamiltonians contain external magnetic fields of different sign. This then implies that the interactions cannot be physically equivalent. ∎

Remarks 1. As noted before, Schonmann has proven that there is no absolutely summable interaction on Ω_1 such that ν_1^+ or ν_1^- is Gibbsian.(3)

2. From the point of view of the cluster expansion ν_2^+ behaves like ν_1^+, since in both cases there are infinitely many contours of different length, all with the same weight, which make the expansion diverge. Therefore we expect that neither ν_2^+ nor, by similar reasoning, ν_2^- is a Gibbs measure.

3. Note that the fact that ν_l^+ and ν_l^-, i.e., the decimations for (the non-Gibbsian) ν_1^+ and ν_1^- onto $l\mathbb{Z}$, are Gibbsian within a certain temperature range is similar to the situation encountered in ref. 5.

4. Because the interactions of ν_l^+ and ν_l^- are not the same, we can conclude that the projection of any mixture $\mu = \lambda\mu^+ + (1-\lambda)\mu^-$, $0 < \lambda < 1$, onto $l\mathbb{Z}$ is non-Gibbsian (see ref. 2, Corollary 4.13)

ACKNOWLEDGMENTS

We have benefited from discussions with A. C. D. van Enter. We also thank C. Maes and M. Winnink for general background discussions. J. L. acknowledges the hospitality of the Instituut voor Theoretische Fysica, Katholieke Universiteit Leuven. K. V. V. is grateful for the hospitality of the Instituut voor Theoretische Natuurkunde, Rijksuniversiteit Groningen.

J. L. is financially supported jointly by the Stichting Fundamenteel Onderzoek der Materia and the Stichting Mathematish Centrum. K. V. V. is an Aspirant NFWO Belgium.

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Communicated by J. L. Lebowitz